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Guaranteed Estimates of the Domain of Attraction for a Class of Hybrid Systems

Chuen Kit Luk, Graziano Chesi and Dongkun Han

Abstract—This paper addresses the estimation of the domain of attraction for a class of hybrid nonlinear systems where the state space is partitioned into several regions. Each region is described by polynomial inequalities, and one of these regions is the complement of the union of all the others in order to ensure complete cover of the state space. The system dynamics is defined on each region independently from the others by polynomial functions. The problem of computing the largest sublevel set of a Lyapunov function included in the domain of attraction is considered. An approach is proposed for addressing this problem based on linear matrix inequalities (LMIs), which provides a lower bound of the sought estimate by establishing negativity of the Lyapunov function derivative on each region. Moreover, a sufficient and necessary condition is provided for establishing optimality of the found lower bound. The results are illustrated by some numerical examples.

I. INTRODUCTION

It is well-known that the domain of attraction of an equilibrium point plays a key role in control systems, from both theoretical and practical viewpoints. Indeed, besides classical examples in electrical systems and mechanical systems, recent studies have shown the importance of investigating the domain of attraction in other fields such as biology, in particular for analyzing the tumor growth dynamics and possibly design a strategy for cancer treatment, and ecology, in particular for characterizing resilience. See e.g. [6], [8]–[10] and references therein.

Several methodologies have been proposed in the literature for estimating the domain of attraction of an equilibrium point. Generally, these methodologies are based on the use of Lyapunov functions and provide inner estimates of the domain of attraction in the form of sublevel sets of a Lyapunov function. Such estimates are established numerically by certifying that they are invariant sets.

For the case of polynomial nonlinear systems, numerous contributions have been proposed to address this step. A group of these contributions is based on convex optimization problems with LMIs, see e.g. [3]–[5], [7], [14]. Another group of contributions makes use of Chebyshev points to establish whether a polynomial is positive, see e.g. [15]. The estimation of the domain of attraction is considered in this paper for hybrid nonlinear systems. The contributions in this area include [1], which aims to obtain estimates in the form of a polytope, and [11], which investigates the use of maximal Lyapunov functions proposed in [16].

This paper addresses the estimation of the domain of attraction for a class of hybrid nonlinear systems where the

state space is partitioned into several regions. Each region is described by polynomial inequalities, and one of these regions is the complement of the union of all the others in order to ensure complete cover of the state space. The system dynamics is defined on each region independently from the others by polynomial functions. The problem of computing the largest sublevel set of a Lyapunov function included in the domain of attraction is considered. An approach is proposed for addressing this problem based on LMIs, which provides a lower bound of the sought estimate by establishing negativity of the Lyapunov function derivative on each region. Moreover, a sufficient and necessary condition is provided for establishing optimality of the found lower bound. The results are illustrated by some numerical examples.

The paper is organized as follows. Section II introduces some preliminaries. Section III describes the estimation of the domain of attraction with fixed Lyapunov functions. Section IV investigates the optimality of the found estimates. Section V presents two illustrative examples. Lastly, Section VI concludes the paper with some final remarks.

II. PRELIMINARIES

A. Problem Formulation

Notation:

- \mathbb{R} : Space of real numbers;
- \mathbb{R}_0 : $\mathbb{R} \setminus \{0\}$;
- 0_n : $n \times 1$ null vector;
- A' : Transpose of A ;
- $A > 0$ ($A \geq 0$): positive definite (semidefinite) matrix;
- $\det(A)$: Determinant of A ;
- s.t.: subject to.

Let us consider a continuous-time hybrid nonlinear systems in the form

$$\begin{cases} \dot{x}(t) &= f(x(t)) \\ x(0) &= x_{init} \end{cases} \quad (1)$$

where $t \in \mathbb{R}$ is the time, $x \in \mathbb{R}^n$ is the state, and $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by

$$f(x) = f_i(x) \text{ if } x \in \mathcal{X}_i, i = 1, \dots, N \quad (2)$$

where $f_i: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $i = 1, \dots, N$, is a polynomial function, and $\mathcal{X}_i \subseteq \mathbb{R}^n$ is defined by

$$\begin{aligned} \mathcal{X}_i &= \{x \in \mathbb{R}^n : z_i(x) \geq 0\}, i = 1, \dots, N-1 \\ \mathcal{X}_N &= \mathbb{R}^n \setminus \mathcal{X}_1 \setminus \dots \setminus \mathcal{X}_{N-1} \end{aligned} \quad (3)$$

where $z_i: \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, 2, \dots, N-1$, is a polynomial. In general cases, $\mathcal{X}_1, \dots, \mathcal{X}_{N-1}$ are closed and \mathcal{X}_N is open.

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For simplicity of description of the proposed approach in the next sections, we define each region \mathcal{X}_i using one polynomial $z_i(x)$ only. We assume that

$$i \neq j \Rightarrow \mathcal{X}_i \cap \mathcal{X}_j = \emptyset \quad (4)$$

that means there are no overlapped regions for both subsystems to be active. We also assume that

$$f(0_n) = 0_n \quad (5)$$

and that the origin is the equilibrium point of interest.

The domain of attraction of the origin of (1) is the set of initial conditions for which the system converges to the origin, i.e.

$$\mathcal{D} = \left\{ x_{init} \in \mathbb{R}^n : \lim_{t \rightarrow +\infty} x(t) = 0_n \right\} \quad (6)$$

where x_{init} is the initial condition of (1).

Let $v : \mathbb{R}^n \rightarrow \mathbb{R}$ be a radially unbounded and positive definite function. We say that $v(x)$ is a Lyapunov function for (1) if

$$\exists \delta > 0 : \dot{v}(x) < 0 \quad \forall x : 0 < \|x\| < \delta \quad (7)$$

where

$$\dot{v}(x) = \nabla v(x)' f(x). \quad (8)$$

The sublevel set $\mathcal{V}(c)$ of $v(x)$ is defined as

$$\mathcal{V}(c) = \{x \in \mathbb{R}^n : v(x) \leq c\}. \quad (9)$$

We have that $\mathcal{V}(c)$ is an estimate of \mathcal{D} if

$$\dot{v}(x) < 0 \quad \forall x \in \mathcal{V}(c) \setminus \{0_n\}. \quad (10)$$

The largest estimate of \mathcal{D} provided by $v(x)$ is the set $\mathcal{V}(c^*)$ where c^* is defined by

$$\begin{aligned} c^* &= \sup_c c \\ \text{s.t. (10) holds.} \end{aligned} \quad (11)$$

B. SOS Polynomials

Let $p(x)$, $x \in \mathbb{R}^n$, be a polynomial. Then, $p(x)$ is said to be a sum of squares of polynomials (SOS) if there exist polynomials $p_1(x), p_2(x), \dots$ such that

$$p(x) = \sum_{i=1} p_i(x)^2. \quad (12)$$

A reason why SOS polynomials are useful is because they are guaranteed to be nonnegative. Another reason is because one can establish whether a polynomial is SOS through a convex optimization problem.

Indeed, let m be the smallest integer such that the degree of $p(x)$ is not greater than $2m$. Then, $p(x)$ can be expressed as

$$p(x) = b(x)'(P + L(\alpha))b(x) \quad (13)$$

where $b(x)$ is a vector whose entries generate a basis for the polynomials in x of degree m , $P = P'$ is a symmetric matrix satisfying

$$p(x) = b(x)'Pb(x) \quad (14)$$

and $L(\alpha)$ is a linear parametrization of the linear subspace

$$\mathcal{L} = \{L = L' : b(x)'Lb(x) = 0\} \quad (15)$$

where α is a free vector with size equal to the dimension of \mathcal{L} . The representation (13) is known as Gram matrix method and square matrix representation (SMR). It turns out that $p(x)$ is SOS if and only if there exists α satisfying the LMI

$$P + L(\alpha) \geq 0. \quad (16)$$

See e.g. [2] and references therein for details about SOS polynomials.

III. COMPUTING ESTIMATES

In the literature several methods have been provided for addressing the computation of c^* in the case of non-hybrid systems, in particular systems where the dynamics are defined by a unique function at any point of the state space. By using such methods, one could get a lower bound of c^* in the case of hybrid systems as considered in this paper. In fact, one could simply ignore the dependence of the system dynamics on the partitions of the state space, and require that the Lyapunov function derivative is negative within a sublevel set of the Lyapunov function for all possible system dynamics. This would provide the lower bound

$$\begin{aligned} \tilde{c} &= \sup_c c \\ \text{s.t. } \dot{v}_i(x) &< 0 \quad \forall x \in \mathcal{V}(c) \setminus \{0_n\} \quad \forall i = 1, \dots, N \end{aligned} \quad (17)$$

where

$$\dot{v}_i(x) = \nabla v(x)' f_i(x). \quad (18)$$

Unfortunately, this lower bound could be very conservative even in the case of very simple systems as shown by the following example.

Example: Let us consider (1) with $x \in \mathbb{R}^2$, $N = 2$ and

$$f_1(x) = \begin{pmatrix} -x_1^2 \\ -x_2 \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}, \quad z_1(x) = x_1. \quad (19)$$

It follows that

$$\mathcal{X}_1 = \{x \in \mathbb{R}^2 : x_1 \geq 0\}, \quad \mathcal{X}_2 = \{x \in \mathbb{R}^2 : x_1 < 0\}. \quad (20)$$

It is straightforward to see that the origin is a globally asymptotically stable equilibrium point, and hence $\mathcal{D} = \mathbb{R}^2$. In particular, by choosing $v(x) = x_1^2 + x_2^2$, one has that $c^* = +\infty$. However, it is also easy to see that $\tilde{c} = 0$ since the origin is an unstable equilibrium point for the subsystem obtained with $f_1(x)$ only. ■

In order to cope with this problem, one should require that the Lyapunov function derivative corresponding to a system dynamics is negative within the portion of a sublevel set of the Lyapunov function where such a dynamics is active. This idea is exploited in the following result which provides a sufficient condition for establishing whether a sublevel set of a Lyapunov function is included in the

domain of attraction of the origin.

Theorem 1: Let $v : \mathbb{R}^n \rightarrow \mathbb{R}$ be a radially unbounded and positive definite function. Let c be a positive real number, and let i , $1 \leq i \leq N$, be an integer. If $i \leq N - 1$, suppose that there exist functions $q_i, s_i : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\left. \begin{aligned} \dot{v}_i(x) + q_i(x)(c - v(x)) + s_i(x)z_i(x) &< 0 \\ q_i(x) &> 0 \\ s_i(x) &\geq 0 \end{aligned} \right\} \forall x \in \mathbb{R}_0^n \quad (21)$$

while, if $i = N$, suppose that there exist functions $q_i, t_1, \dots, t_{N-1} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\left. \begin{aligned} \dot{v}_N(x) + q_N(x)(c - v(x)) + \\ - \sum_{j=1}^{N-1} t_j(x)z_j(x) &< 0 \\ q_i(x) &> 0 \\ t_j(x) &\geq 0 \end{aligned} \right\} \forall j = 1, \dots, N-1 \quad \forall x \in \mathbb{R}_0^n \quad (22)$$

Then,

$$\dot{v}_i(x) < 0 \quad \forall x \in \mathcal{V}_i(c) \setminus \{0_n\} \quad (23)$$

where

$$\mathcal{V}_i(c) = \mathcal{V}(c) \cap \mathcal{X}_i. \quad (24)$$

Moreover, if for all $i = 1, \dots, N - 1$ there exist functions $q_i, s_i : \mathbb{R}^n \rightarrow \mathbb{R}$ such that (21) holds, and if for $i = N$ there exist functions $q_i, t_1, \dots, t_{N-1} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that (22) holds, then $v(x)$ is a Lyapunov function for the origin, and $\mathcal{V}(c) \subseteq \mathcal{D}$.

Proof: Let i , $1 \leq i \leq N$, be an integer. Suppose that $\mathcal{V}_i(c) \setminus \{0_n\}$ is nonempty, and let us consider any x in such a set. If $i \leq N - 1$, suppose that (21) holds. From the first inequality in (21) one gets

$$\begin{aligned} 0 &> \dot{v}_i(x) + q_i(x)(c - v(x)) + s_i(x)z_i(x) \\ &\geq \dot{v}_i(x) \end{aligned}$$

since $q_i(x) > 0$ from (21), $c - v(x) \geq 0$ since $x \in \mathcal{V}(c) \setminus \{0_n\}$, $s_i(x) \geq 0$ from (21), and $z_i(x) \geq 0$ since $x \in \mathcal{X}_i$. If $i = N$, suppose that (22) holds. Repeating the observations made for the previous case, we similarly obtain that $\dot{v}_i(x) < 0$, i.e. (23) holds.

Next, suppose that (21) holds for all $i = 1, \dots, N - 1$ and that (22) holds for $i = N$. Since

$$\bigcup_{i=1}^N \mathcal{X}_i = \mathbb{R}^n$$

it follows that

$$\bigcup_{i=1}^N \mathcal{V}_i(c) = \mathcal{V}(c)$$

and, hence, that (10) holds. Therefore, $\mathcal{V}(c) \subseteq \mathcal{D}$. Moreover, since c is positive and $v(x)$ is positive definite, this implies that

$$\exists \delta > 0 : \{x \in \mathbb{R}^n : \|x\| < \delta\} \subset \mathcal{V}(c)$$

i.e. $v(x)$ is a Lyapunov function for the origin. ■

Theorem 1 provides a condition for establishing whether $\mathcal{V}(c)$ is included in \mathcal{D} through the existence of suitable

functions satisfying the inequalities (21) and (22). Whenever $v(x)$, $q_i(x)$, $s_i(x)$ and $t_j(x)$ are polynomial functions, this condition exploits Stengle's Positivstellensatz [12] and can be checked through LMIs by using SOS polynomials as described in Section II-B.

Theorem 1 can be used to compute a lower bound of c^* . First of all, let us define

$$c_i^* = \sup_c c \quad \text{s.t. (23) holds.} \quad (25)$$

It is easy to see that

$$c^* = \min_{i=1, \dots, N} c_i^*. \quad (26)$$

By using Theorem 1, one can obtain a lower bound of c_i^* as

$$\begin{aligned} \hat{c}_i^* &= \sup_c c \\ \text{s.t. } &\begin{cases} (21) \text{ holds} & \text{if } i \leq N - 1 \\ (22) \text{ holds} & \text{if } i = N. \end{cases} \end{aligned} \quad (27)$$

Such a lower bound is computed through a bisection algorithm on the scalar c where, for each fixed value of c , the condition of Theorem 1 is investigated by using LMIs for fixed degrees of the variable polynomials $q_i(x)$, $s_i(x)$ and $t_i(x)$. From the lower bounds \hat{c}_i^* of c_i^* we can define the lower bound \hat{c}^* of c^* as

$$\hat{c}^* = \min_{i=1, \dots, N} \hat{c}_i^*. \quad (28)$$

Indeed, from Theorem 1 one has

$$\hat{c}_i^* \leq c_i^* \quad \forall i = 1, \dots, N \quad (29)$$

and, hence,

$$\hat{c}^* \leq c^*. \quad (30)$$

IV. OPTIMALITY OF THE ESTIMATES

In the previous section we have shown how the lower bound \hat{c}^* of c^* can be computed. A natural question is whether this lower bound is optimal, i.e. $\hat{c}^* = c^*$. The next result provides a sufficient and necessary condition for answering to this question.

Theorem 2: Without loss of generality, let us suppose that $\hat{c}^* < \infty$. Let us define the set

$$\mathcal{I}^* = \{i = 1, \dots, N : \hat{c}_i^* = \hat{c}^*\}. \quad (31)$$

Then, $\hat{c}^* = c^*$ if and only if there exist $i \in \mathcal{I}^*$ and $x \in \mathbb{R}^n$ such that

$$\begin{cases} \dot{v}_i(x) = 0 \\ v(x) = \hat{c}_i^* \\ x \in \mathcal{X}_i \end{cases} \quad (32)$$

and, consequently, that

$$\dot{v}_i(x) + q_i^*(x)(\hat{c}_i^* - v(x)) + s_i^*(x)z_i(x) = 0 \quad (33)$$

if $i \leq N - 1$, or

$$\dot{v}_N(x) + q_N^*(x)(\hat{c}_i^* - v(x)) - \sum_{j=1}^{N-1} t_j^*(x)z_j(x) = 0 \quad (34)$$

if $i = N$.

Proof: (Necessity) Let us suppose that $\hat{c}^* = c^*$. Let x^* be an optimal point of (11), i.e. such that

$$\begin{aligned}\dot{v}(x^*) &= 0 \\ v(x^*) &= c^*.\end{aligned}$$

Obviously, such a point exists because $c^* = \hat{c}^* < \infty$ and because, supposing for contradiction that it does not, one would obtain that c^* is not the solution of (11). Let i be the integer in $[1, n]$ such that

$$x^* \in \mathcal{X}_i.$$

It follows that $c_i^* = c^*$, which implies that $\hat{c}_i^* = \hat{c}^*$. This means that

$$i \in \mathcal{I}^*.$$

Let us also observe that, for definition of $\dot{v}(x)$, one has that

$$\dot{v}_i(x^*) = 0$$

and hence (32) holds. Next, let us suppose that $i \leq N - 1$. For definition of \hat{c}_i^* , one has that

$$\dot{v}_i(x) + q_i^*(x)(\hat{c}_i^* - v(x)) + s_i^*(x)z_i(x) \leq 0 \quad \forall x \in \mathbb{R}^n.$$

Hence, for $x = x^*$ one obtains that

$$s_i^*(x^*)z_i(x^*) \leq 0.$$

Since $s_i^*(x) \geq 0$ for all $x \in \mathbb{R}^n$ and since $z_i(x^*) \geq 0$, it follows that

$$s_i^*(x^*)z_i(x^*) = 0$$

and hence that

$$\dot{v}_i(x^*) + q_i^*(x^*)(\hat{c}_i^* - v(x^*)) + s_i^*(x^*)z_i(x^*) = 0$$

which proves (33). Similarly, if $i = N$, one obtains that

$$\dot{v}_N(x^*) + q_N^*(x^*)(\hat{c}_i^* - v(x^*)) - \sum_{j=1}^{N-1} t_j^*(x^*)z_j(x^*) = 0$$

which proves (34).

(Sufficiency) Let us suppose that there exist $i \in \mathcal{I}^*$ and $x \in \mathbb{R}^n$ such that (32) holds. This means that \hat{c}_i^* is optimal, i.e.

$$\hat{c}_i^* = c_i^*.$$

Since $i \in \mathcal{I}^*$ it follows that

$$\hat{c}_i^* = \hat{c}^*.$$

Let us also observe that

$$\hat{c}^* \leq c^* \leq c_i^*$$

which clearly implies that $\hat{c}^* = c^*$. ■

Theorem 2 provides a sufficient and necessary condition for establishing whether the lower bound \hat{c}^* is optimal. In particular, this condition requires to check the existence of $x \in \mathbb{R}^n$ satisfying (32). This can be done by observing that such x must also satisfy (33) if $i \leq N - 1$ or (34) if $i = N$. Determining the zeros of the left hand sides of (33)–(34) can be addressed by solving linear algebra operations as discussed in [3].

V. EXAMPLES

Here we present two illustrative examples of the proposed approach. The computations are done in Matlab using the toolbox SeDuMi [13]. The degrees of the polynomial functions $q_i(x)$, $s_i(x)$ and $t_j(x)$ are chosen as the largest degrees for which the left hand side of the first inequalities in (21) and (22) have the minimum even degree.

A. Example 1

Let us consider the hybrid nonlinear system (1) with $N = 2$ and

$$\begin{aligned}f_1(x) &= \begin{pmatrix} -x_1 + 2x_1^3x_2 \\ -x_2 - x_1x_2^2 \end{pmatrix} \\ f_2(x) &= \begin{pmatrix} -x_1 + x_1x_2^2 \\ -x_2 + x_1x_2 + x_1^3 \end{pmatrix} \\ z_1(x) &= x_1.\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{X}_1 &= \{x \in \mathbb{R}^2 : x_1 \geq 0\} \\ \mathcal{X}_2 &= \{x \in \mathbb{R}^2 : x_1 < 0\}.\end{aligned}$$

We consider the estimation of the domain of attraction of the origin with the Lyapunov function

$$v(x) = x_1^2 + x_2^2.$$

In particular, we want to find the largest estimate provided by $v(x)$, i.e. compute c^* in (11). To this end, let us use Theorem 1. We compute the lower bounds \hat{c}_1^* and \hat{c}_2^* defined in (27), finding

$$\hat{c}_1^* = 1.55, \quad \hat{c}_2^* = 2.64.$$

These lower bounds provide the lower bound \hat{c}^* of c^* defined in (28) and given by

$$\hat{c}^* = 1.55.$$

In order to establish whether this lower bound is optimal, let us use Theorem 2. We have that $\mathcal{I}^* = \{1\}$, and (32) holds with

$$i = 1, \quad x = (1.14, 0.50)'$$

Hence, from Theorem 2 we conclude that \hat{c}^* is optimal, i.e.

$$c^* = \hat{c}^* = 1.55.$$

Figures 1 and 2 show the regions \mathcal{X}_i , the curves $\dot{v}_i(x) = 0$ and the boundaries of the estimates $\mathcal{V}(\hat{c}_i^*)$ for $i = 1, 2$.

B. Example 2

Let us consider the hybrid nonlinear system (1) with $N = 3$ and

$$\begin{aligned}f_1(x) &= \begin{pmatrix} -x_1 \\ -x_1 - x_2 + x_1x_2 \end{pmatrix} \\ f_2(x) &= \begin{pmatrix} -x_1 + x_2 - x_2^2 \\ -x_2 + x_1^3x_2^2 \end{pmatrix} \\ f_3(x) &= \begin{pmatrix} -x_1 - x_2 \\ -x_2 + 2x_1x_2^2 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}z_1(x) &= -x_1^2 - x_2^2 + 2x_1 + 2x_2 - 1 \\ z_2(x) &= -x_1^2 - x_2^2 - 2x_1 - 2x_2 - 1.\end{aligned}$$

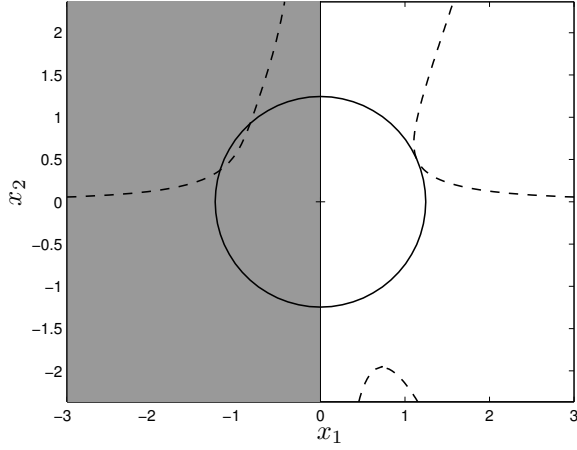


Fig. 1. Example 1: \mathcal{X}_1 (white area), curve $\dot{v}_1(x) = 0$ (dashed lines) and boundary of $\mathcal{V}(\hat{c}_1^*)$ (solid line).

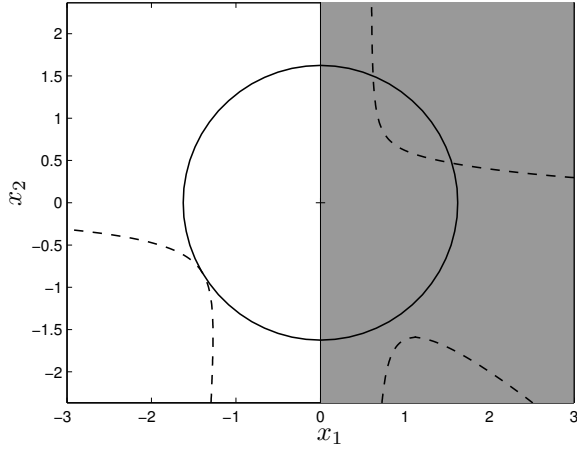


Fig. 2. Example 1: \mathcal{X}_2 (white area), curve $\dot{v}_2(x) = 0$ (dashed lines) and boundary of $\mathcal{V}(\hat{c}_2^*)$ (solid line).

It follows that

$$\begin{aligned}\mathcal{X}_1 &= \{x \in \mathbb{R}^2 : (x_1 + 1)^2 + (x_2 + 1)^2 \leq 1\} \\ \mathcal{X}_2 &= \{x \in \mathbb{R}^2 : (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1\} \\ \mathcal{X}_3 &= \mathbb{R}^2 \setminus (\mathcal{X}_1 \cup \mathcal{X}_2).\end{aligned}$$

We consider the estimation of the domain of attraction of the origin with the Lyapunov function

$$v(x) = x_1^2 + x_2^2 + x_1^4 + x_2^4.$$

In particular, we want to find the largest estimate provided by $v(x)$, i.e. compute c^* in (11). To this end, let us use Theorem 1. We compute the lower bounds \hat{c}_i^* defined in (27), finding

$$\hat{c}_1^* = \infty, \quad \hat{c}_2^* = 7.24, \quad \hat{c}_3^* = 11.71.$$

These lower bounds provide the lower bound \hat{c}^* of c^* defined in (28) and given by

$$\hat{c}^* = 7.24.$$

In order to establish whether this lower bound is optimal, let us use Theorem 2. We have that $\mathcal{I}^* = \{2\}$, and (32) holds with

$$i = 2, \quad x = (1.15, 1.27)'.$$

Hence, from Theorem 2 we conclude that \hat{c}^* is optimal, i.e.

$$c^* = \hat{c}^* = 7.24.$$

Figures 3–5 show the regions \mathcal{X}_i , the curves $\dot{v}_i(x) = 0$ and the boundaries of the estimates $\mathcal{V}(\hat{c}_i^*)$ for $i = 1, 2, 3$.

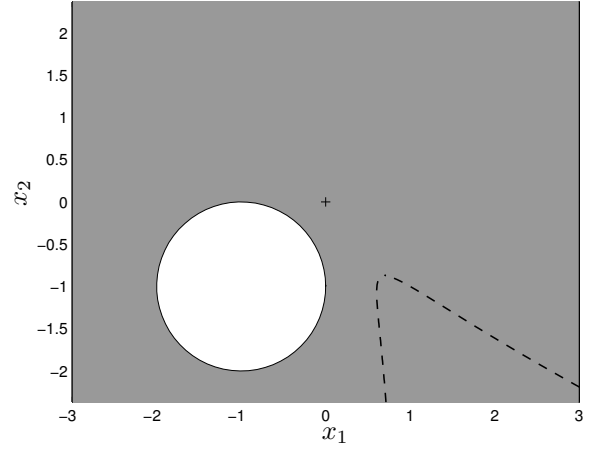


Fig. 3. Example 2: \mathcal{X}_1 (white area) and curve $\dot{v}_1(x) = 0$ (dashed lines). The boundary of $\mathcal{V}(\hat{c}_1^*)$ is at infinity since $\hat{c}_1^* = \infty$ as the curve $\dot{v}_1(x) = 0$ does not enter the region \mathcal{X}_1 .

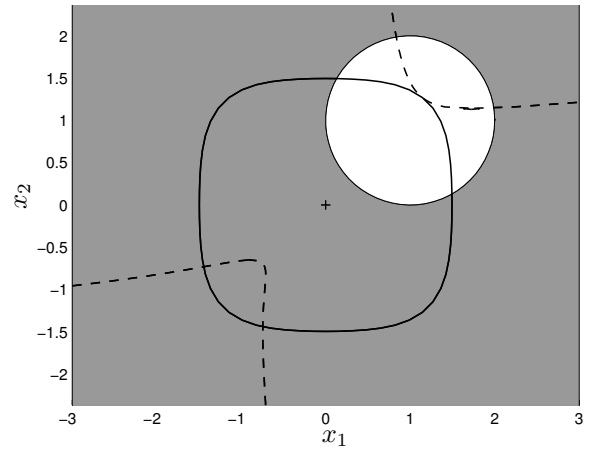


Fig. 4. Example 2: \mathcal{X}_2 (white area), curve $\dot{v}_2(x) = 0$ (dashed lines) and boundary of $\mathcal{V}(\hat{c}_2^*)$ (solid line).

VI. CONCLUSIONS

We have considered the estimation of the domain of attraction for a class of hybrid nonlinear systems where the state space is partitioned into several regions. In particular, we have proposed an approach for computing the largest

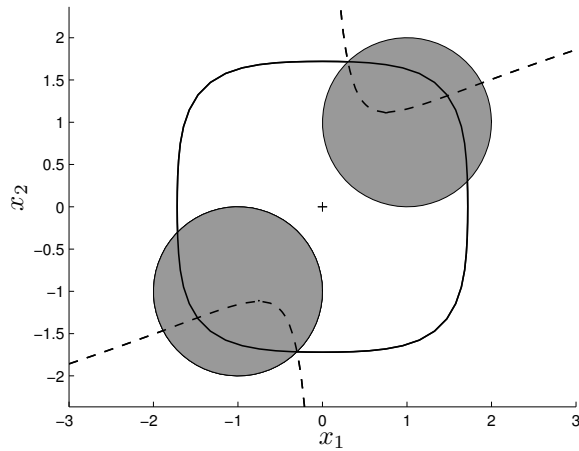


Fig. 5. Example 2: \mathcal{X}_3 (white area), curve $v_3(x) = 0$ (dashed lines) and boundary of $\mathcal{V}(\hat{c}_3^*)$ (solid line).

sublevel set of a Lyapunov function included in the domain of attraction based on LMIs, which provides a lower bound of the sought estimate by establishing negativity of the Lyapunov function derivative on each partition of the state space. We have also provided a sufficient and necessary condition for establishing optimality of the found lower bound.

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